RAMAKRISHNA MISSION VIDYAMANDIRA
(Residential Autonomous College affiliated to University of Calcutta)
B.A./B.Sc. THIRD SEMESTER EXAMINATION(Batch 2019-22), March 2021
SECOND YEAR

Date : 12/03/2021 **MATHEMATICS HONOURS**
Time : 11 am - 1 pm **Paper : MACT 5[CC5]** **Full Marks : 50**

Instructions to the students

- Write your **College Roll No, Year, Subject** & **Paper Number** on the top of the **Answer Script**.
- Write your **Name, College Roll No, Year, Subject** & **Paper Number** on the **text box of your e-mail**.
- Read the instructions given at the beginning of each paper/group/unit carefully.
- Only handwritten (by blue/black pen) answer-scripts will be permitted.
- Try to answer all the questions of a single group/unit at the same place.
- All the pages of your answer script must be numbered serially by hand.
- In the last page of your answer-script, please mention the total number of pages written so that we can verify it with that of the scanned copy of the script sent by you.
- For an easy scanning of the answer script and also for getting better image, students are advised to write the answers on single side and they must give a minimum 1 inch margin at the left side of each paper.
- After the completion of the exam, scan the entire answer script by using Clear Scan: Indy Mobile App or any other Scanner device and make a **single PDF file(Named as your College Roll No)** and send it to [redacted]

Group - A

Answer any 4 questions from question numbers 1 to 6. **[20 marks]**

1. Let V be the vector space of all polynomial functions p from $\mathbb{R}$ into $\mathbb{R}$ which have degree 2 or less: $p(x) = c_0 + c_1x + c_2x^2$. Define three linear functionals on V by $f_1(p) = \int_0^1 p(x)dx$, $f_2(p) = \int_0^2 p(x)dx$ and $f_3(p) = \int_0^{-1} p(x)dx$. Show that $\{f_1, f_2, f_3\}$ is a basis for V^* by exhibiting the basis for V of which it is the dual. [5]

2. Let W be a finite dimensional subspace of an inner product space V, and let E be the orthogonal projection of V on W. Prove that $(E\alpha|\beta) = (\alpha|E\beta)$ for all $\alpha, \beta \in V$. [5]

3. Let V be the subspace of $\mathbb{R}[x]$ of polynomials of degree at most 3. Equip V with the inner product $(f|g) = \int_0^1 f(t)g(t)dt$. Apply the Gram-Schmidt process to the basis $\{1, x, x^2, x^3\}$. [5]

4. For the vector space $\mathbb{R}^3$ and basis $\beta = \{(1,0,1),(1,2,1),(0,0,1)\}$, find the dual basis β^* for V^*. [5]

5. Let V be a complex vector space with an inner product $(\;|\;)$. Prove that $(\alpha|\beta) = \frac{1}{4}\sum\limits_{n=1}^{4} i^n ||\alpha + i^n\beta||^2$ for $\alpha, \beta \in V$. [5]

6. Describe explicitely all inner products of $\mathbb{R}^2$. [5]

Group - B

Answer all the questions. Maximum you can score is 30.

7. Let H be a subgroup of a group G. Define $K = \bigcap\limits_{g\in G} g^{-1}Hg$. Prove that K is normal in G. Also prove that if L is a normal subgroup of G and $L \subseteq H$ then $L \subseteq K$. [2+2]

8. Let G be group of order 8 and let x be an element of G of order 4. Prove that $x^2 \in Z(G)$. [4]

9. Show that $\frac{(\mathbb{C},+)}{(\mathbb{R},+)} \cong (\mathbb{R},+)$. [3]

10. Let R be a ring with 1 and $a \in R$. If $\exists$ a unique $b \in R$ such that $ab = 1$, prove that $ba = 1$. [2]

11. Prove that $(\mathcal{P}(\mathbb{N}), \Delta, \bigcap)$ is a commutative ring with identity. Does the ring contain divisor of zero? What is the characteristic of the ring? (Here $\mathcal{P}(\mathbb{N})$ denotes the power set of $\mathbb{N}$). [3+1+1]

12. Find all units of $(\mathbb{Z}_8, +, \cdot)$. Prove that the units of $\mathbb{Z}_8$ form a noncyclic group with respect to multiplication. [4]

13. Show that the rings $(2\mathbb{Z}, +, \cdot)$ and $(3\mathbb{Z}, +, \cdot)$ are not isomorphic. [3]

14. Find all units of the ring $\mathbb{Z}[i\sqrt{5}]$. [3]

15. Show that $2 - i$ is irreducible in $\mathbb{Z}[i]$. [3]

16. Let $\mathbb{Q}_3 = \{r \in \mathbb{Q} : r = \frac{a}{b}$ and $\gcd(a,b) = 1 \Rightarrow 3$ does not divide $b\}$. Show that $\mathbb{Q}_3$ is a ring with identity with usual addition and multiplication. Show also that all nonunits in $\mathbb{Q}_3$ form a maximal ideal of $\mathbb{Q}_3$. [3+2]